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SOME IMPROVED BOUNDS ON THE NUMBER OF BLOCKS IN BIB DESIGNS

SANPEI KAGEYAMA*
Hiroshima University, Hiroshima 734, Japan
and
S. M. SHAH and C. C. GUJARATHI
Sardar Patel University, Vallabh Vidyanagar, Gujarat, India

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SUMMARY

In a balanced incomplete block (BIB) design with parameters v, b, r, k and λ, Shah [6] obtained an upper bound $b \leqslant (r^2 - 1)/\lambda$ on b. The main purpose of this paper is to improve this inequality and to obtain some upper bounds on the number of blocks in a BIB design under reasonable parametric restrictions.

Introduction

For the number of blocks of a BIB design with parameters v, b, r, k and λ, it is well known that the Fisher inequality $b \geqq v$ holds. On the other hand, Shah [6] gave an upper bound $b \leqq (r^2 - 1)/\lambda$, and it is known that this bound is attained when $r = k = \lambda + 1$. Since the design becomes a trivial design when $r = \lambda + 1$, by relaxing this attaining condition, in this paper the Shah bound is improved in some directions under certain reasonable parametric restrictions. Finally, the Kageyama bound [2] $b \geqq (m^2\lambda + m)/\mu^2$ for a BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ is derived without an assumption of $b = mt$.

Discussions and Results

It is known that the Shah inequality $b \leqq (r^2 - 1)/\lambda$ is attained when

*Prepared while visiting the Department of Statistics, Sardar Patel University.

$r = k$ and $r = \lambda + 1$. In order to improve this inequality, we consider its attaining condition. In this case, the following characterization is given.

LEMMA. *A BIB design with parameters v, b, r, k and λ satisfying $r = \lambda + 1$ is a symmetrical BIB design with parameters $v = b = \lambda + 2$, $r = k = \lambda + 1$ and λ.*

Proof. From a basic relation $\lambda(v - 1) = r(k - 1)$ and the assumption $r = \lambda + 1$, we get $v = (\lambda + 1)(k - 1)/\lambda + 1$, which, from $(\lambda, \lambda + 1) = 1$, implies that there exists a positive integer l such that $k - 1 = l\lambda$. Since $r \geq k$, we get $l = 1$, i.e., $r = k = \lambda + 1$. Then $v = \lambda + 2$. Thus, we have the required result.

Under the observation in the lemma, we shall give below some theorems on upper bounds for the number of blocks in a BIB design. In what follows, a symbol $[x]$ means the largest integer $< x$, where x is not an integer.

THEOREM 2.1. *In a BIB design with parameters v, b, r, k and λ, if $r \neq \lambda + 1$, then $b \leqq (r^2 - 2)/\lambda$, with the equality holding when $r = k = \lambda + 2$.*

Proof. From a basic relation $\lambda(v - 1) = r(k - 1)$ of a BIB design, we get

$$rk - \lambda v = r - \lambda. \tag{2.1}$$

Since $r - \lambda \geqq 1$ in general and $r \neq \lambda + 1$, we get $r - \lambda \geqq 2$, which, from (2.1), yields $rk - \lambda v \geqq 2$. This, after multiplying by r and simplifying, yields

$$b \leqq r^2/\lambda - 2r/(\lambda k). \tag{2.2}$$

Since $r \geqq k$ in general, $b \leqq (r^2 - 2)/\lambda$. It is obvious that $b = (r^2 - 2)/\lambda$ is attained when $r = k = \lambda + 2$.

As a refinement of Theorem 2.1, we get the following.

PROPOSITION 2.2. In a non-symmetrical BIB design with parameters v, b, r, k and λ,

(i) $b \leqq (r^2 - 2)/\lambda - 1$ if $(r^2 - 2)/\lambda =$ an integer;
(ii) $b \leqq [(r^2 - 2)/\lambda]$ if $(r^2 - 2)/\lambda \neq$ an integer.

Proof. Since $r > k$, the lemma and (2.2) yield $b < (r^2 - 2)/\lambda$, which shows the required result.

THEOREM 2.3. *In a BIB design with parameters v, b, r, k and λ, if $k \leqq v/n$ for an integer $n \geqq 2$, then $b \leqq (r^2 - 1)/\lambda - (n - 1)$, with the equality holding when $r = k = n\lambda + 1$.*

Proof. It follows that for $n \geqq 2$,

$$\begin{aligned} k \leqq v/n &\Leftrightarrow nk - 1 \leqq v - 1 \\ &\Leftrightarrow (n-1)\, rk + r(k-1) \leqq r(v-1) \\ &\Leftrightarrow r \geqq \{\lambda(v-1) + rk(n-1)\}/(v-1) \\ &\Leftrightarrow r - n\lambda \geqq r(n-1)/(v-1) > 0 \\ &\Leftrightarrow r - n\lambda \geqq 1. \end{aligned} \tag{2.3}$$

From (2.1) and (2.3), we get $rk - \lambda v \geqq (n - 1)\lambda + 1$, which, upon multiplying by r and simplifying, yields $b \leqq r^2/\lambda - (n-1)r/k - r/(k\lambda)$. Since $r \geqq k$ in general,

$$b \leqq (r^2 - 1)/\lambda - (n - 1). \tag{2.4}$$

It is clear that this bound is attained when $r = k = n\lambda + 1$.

The following is the refinement of Theorem 2.3.

PROPOSITION 2.4. In a non- symmetrical BIB design with parameters v, b, r, k and λ, if $k \leqq v/n$ for $n \geqslant 2$, then

(i) $b \leqq (r^2 - 1)/\lambda - n$ when $(r^2 - 1)/\lambda =$ an integer;

(ii) $b \leqq [(r^2 - 1)/\lambda] - (n - 1)$ when $(r^2 - 1)/\lambda \neq$ an integer.

Proof. Since $r > k$, from (2.4) we get $b < (r^2 - 1)/\lambda - (n - 1)$ which yields the required result.

For a BIB design or its complementary design, one can always find that $k \leqq v/2$. Hence, without loss of generality, consider a BIB design satisfying $k \leqq v/2$. In this case, Theorem 2.3 and Proposition 2.4 yield the following when $n = 2$.

COROLLARY 2.4.1. *In a BIB design with parameters v, b, r, k and λ, if $k \leqq v/2$, then $b \leqq (r^2 - 1)/\lambda - 1$. Furthermore, for a non-symmetrical BIB design, $b \leqq (r^2 - 1)/\lambda - 2$ if $(r^2 - 1)/\lambda =$ an integer: and $b \leqq [(r^2 - 1)/\lambda] - 1$ if $(r^2 - 1)/\lambda \neq$ an integer.*

Remark 2.1. Theorems 2.1 and 2.3, and Propositions 2.2 and 2.4 give improvements for the Shah inequality $b \leqq (r^2 - 1)/\lambda$ under some parametric restrictions which are reasonable.

Kageyama [1] has shown that in a BIB design with parameters $v = nk$, b, r, k and λ,

$$b > v + r - 2 \Leftrightarrow r \geqq \lambda + 2k. \tag{2.5}$$

This gives us the following.

THEOREM 2.5. *In a BIB design with parameters* $v = nk$, b, r, k *and* λ, *if* $b > v + r - 1$, *then* $b \leqq r(r-2)/\lambda$ *and vice versa. The inequality is attained if and only if* $r = \lambda + 2k$.

Proof. From (2.1) and (2.5), we get

$$\begin{aligned}(r - n\lambda)\, k \geqq 2k &\Leftrightarrow r - 2 \geqq n\lambda\\ &\Leftrightarrow r\,(r - 2) \geqq nr\lambda\\ &\Leftrightarrow b \leqq r(r - 2)/\lambda.\end{aligned}$$

The converse is obvious. An attaining case of this inequality follows from (2.5).

Remark 2.2. Theorem 2.5 gives an improvement of an inequality $b \leqq r(r - 1)/\lambda$ due to Shah [6] under an assumption.

Theorem 2.5 yields immediately the following in terms of practical concepts.

COROLLARY 2.5.1. *In a resolvable BIB design which is not affine resolvable,* $b \leqq r(r - 2)/\lambda$, *the equality holding if and only if* $r - \lambda = 2k$.

COROLLARY 2.5.2. *In a resolvable BIB design which is not affine resolvable, if* $r = \lambda + 2k$, *then* $v = (r - 2)(r - \lambda)/(2\lambda)$, $b = r(r - 2)/\lambda$, $k = (r - \lambda)/2$.

COROLLARY 2.5.3. *In a BIB design with parameters* $v = nk$, b, r, k, λ *satisfying* $b > v + r - 1$ *and* $r - \lambda \neq 2k$, *we have* $b \leqq r(r - 3)/\lambda$, *with the equality holding if and only if* $r - n\lambda = 3$ ($\Leftrightarrow r - \lambda = 3k$).

Example 2.2. Consider a resolvable (unreduced) BIB design with parameters $v = 8$, $b = 28$, $r = 7$, $k = 2$ and $\lambda = 1$ (cf. Raghavarao [5]). The bound $b \leqq r(r - 3)/\lambda$ is attained for this BIB design.

Ramark 2.3. Under the same situation as in Corollary 2.4.1, Kageyama [3] gave some improved lower bounds on the number of blocks in a BIB design.

Shah [7] has shown that for a BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ,

$$b \gtreqless v + r - 1 \Leftrightarrow \mu r - m\lambda \gtreqless \mu, \tag{2.6}$$

respectively. This gives the following.

THEOREM 2.6. *In a BIB design with parameters* v, $b = mt$, $r = \mu t$, k *and* λ,

$$b \gtreqless v + r - 1 \Leftrightarrow b \gtreqless (m^2\lambda + m\mu)/\mu^2,$$

respectively.

Proof. From (2.6), we have

$$b \gtreqless v + r - 1 \Leftrightarrow \mu r m \gtreqless m^2\lambda + m\mu$$
$$\Leftrightarrow mt \gtreqless (m^2\lambda + m\mu)/\mu^2$$
$$\Leftrightarrow b \gtreqless (m^2\lambda + m\mu)/\mu^2.$$

Remark 2.4. The inequalities in Theorem 2.6 are attained at the same time. When $\mu = 1$, a case $b \geqq v + r - 1$ only has a meaning.

Two expressions in Theorem 2.6 have some meaning, as the following examples show.

Example 2.2. Consider a BIB design with parameters $v = 6$, $b = 20$, $r = 10$, $k = 3$ and $\lambda = 4$ (cf. Takeuchi [8]), having $\mu = 2$, $m = 4$ and $t = 5$. In this case, $v + r - 1 = 15$, $(m^2\lambda + m\mu)/\mu^2 = 18$ and $b = 20$.

Example 2.3. Consider a BIB design with parameters $v = 49$, $b = 56$, $r = 24$, $k = 21$ and $\lambda = 10$ (cf. Kageyama and Mohan [4]), having $\mu = 3$, $m = 7$ and $t = 8$. In this case, $v + r - 1 = 72$, $(m^2\lambda + m\mu)/\mu^2 = 56.28$ and $b = 56$. The second bound is actually attained, since the number of blocks is an integer.

As a result of these observations, we have

COROLLARY 2.6.1. *In a BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ,*

$$b \gtreqless v + r - 1 \Leftrightarrow b \gtreqless \frac{m^2\lambda + m\mu}{\mu^2} \Leftrightarrow \mu r - m\lambda \gtreqless \mu \Leftrightarrow r \gtreqless k + \lambda,$$

respectively.

Kageyama [2] derived a bound $b \geqslant (m^2\lambda + m)/\mu^2$ for a BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ. This bound can also be given by relaxing a condition $b = mt$ as follows.

THEOREM 2.7. *In a BIB design with parameters v, b, $r = \mu t$, k and λ, $b \geqq (\lambda m^2 + m)/\mu^2$, where m is the largest integer $\leqq b/t$.*

Proof. From (2.1) and the basic relation $bk = vr$, we get $(r^2 - \lambda b)k = r(r - \lambda) > 0$, which yields

$$r^2 - \lambda b \geqq 1. \tag{2.7}$$

Let $b = mt + i$, $0 \leqq i \leqq t - 1$. Substituting this in (2.7) we get

$$(\mu r - \lambda m)\, t \geqq 1 + \lambda i \geqslant 1$$
$$\Rightarrow \mu r - \lambda m \geqslant 1$$
$$\Leftrightarrow (\mu r - \lambda m)\, m \geqslant m$$

$\Leftrightarrow \mu rm \geqq \lambda m^2 + m$

$\Leftrightarrow \mu rm/\mu^2 \geqq (\lambda m^2 + m)/\mu^2$

$\Leftrightarrow mt \geqq (\lambda m^2 + m)/\mu^2$ since $r = \mu t$

$\Rightarrow b \geqq (\lambda m^2 + m)/\mu^2$ since $m \leqq b/t$.

Remark 2.5. Taking $i = 0$ in $b = mt + i$, we get the result of Kageyama [2].

3. Concluding Remark

This type of improvements for known inequalities can always be given if one puts additional parametric restrictions on a BIB design. Artificial restrictions are infinite in number so is the number of new bounds. However, the conditions of non-symmetry ($b \neq v$), $k \leqq v/2$, or $r \neq +1$ on the designs discussed here are very simple and reasonable restrictions. It appears that almost all of BIB designs satisfy the above restrictions. That is why we choose these restrictions as a motivation of this paper, from a point of view of an improvement of an inequality.

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